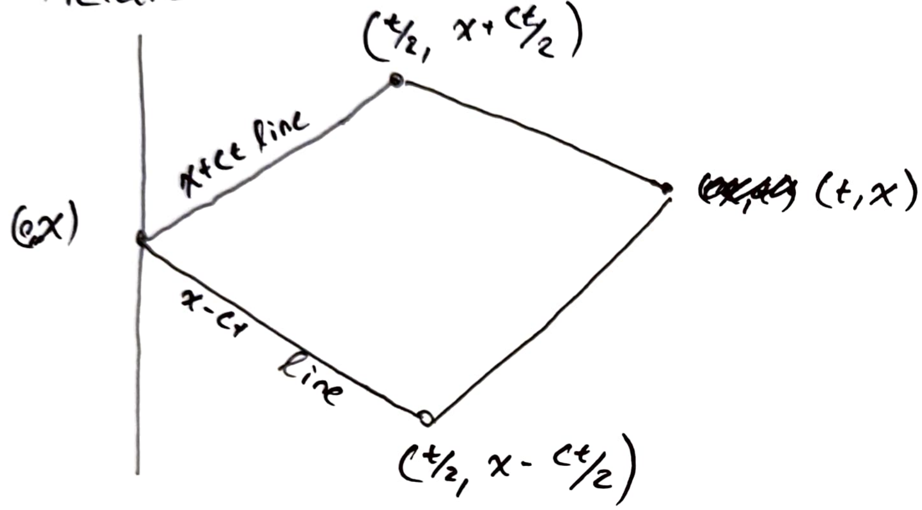
· HW 2 Solutions

Problem 1

Picture:



$$u(t,x) = \frac{1}{2}\left[u(0,x+ct) + u(0,x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \partial_t u(0,\tau)\,d\tau$$

$$u(t/2, x + ct/2) = \frac{1}{2}\left[u(0,x+ct) + u(0,x)\right] + \frac{1}{2c}\int_{x}^{x+ct} \partial_t u(0,\tau)\,d\tau$$

$$u(t/2, x - ct/2) = \frac{1}{2}\left[u(0,x) + u(0,x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x} \partial_t u(0,\tau)\,d\tau$$

$$\Rightarrow \quad u(t,x) = u(t/2, x + ct/2) + u(t/2, x - ct/2) - u(0,x)$$

Problem 2

$$u(t,x) = \frac{1}{2} \iint_{D_{t,x}} f(\tau, y)\, d\tau dy$$

$$= \frac{1}{2} \iint_{D_{t,x}} \cos(\omega\tau) \sin(\omega_1 y)\, d\tau dy$$

From here, the computation is as in lecture.

Problem 3

If $u(t,x) = e^{-at/2} w(t,x)$, $\quad w(t,x) = e^{at/2} u(t,x)$

and

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Leftrightarrow \quad e^{-at/2}\left(\frac{\partial^2 w}{\partial t^2} - c^2 \partial_x^2 w\right) = 0$$

$$w(0,x) = u(0,x) = g(x)$$

$$\partial_t w(t,x) = \frac{a}{2} e^{at/2} u(t,x) + e^{at/2}\partial_t u(t,x)$$

$$\text{so } \partial_t w(0,x) = \frac{a}{2} g(x) + h(x)$$

$$w(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{a}{2} g(\tau) + h(\tau)\, d\tau$$

$$u(t,x) = e^{-at/2}\left[\frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{a}{2} g(\tau) + h(\tau)\, d\tau\right]$$

Problem ~~3~~ 4

A.) $\partial_t \int_U u\bar{u}\,dx = \int_U \frac{\partial u}{\partial t}\bar{u} + u\frac{\partial \bar{u}}{\partial t}\,dx$

$= \int_U (i\Delta u)\bar{u} - iu\Delta\bar{u}\,dx = i\int_U -u\Delta\bar{u} + \bar{u}\Delta u\,dx$

$= i\int_{\partial U} -u\frac{\partial \bar{u}}{\partial \eta} + \bar{u}\frac{\partial u}{\partial \eta}\,dx = 0$ b/c $u|_{\partial U} = 0 = \bar{u}|_{\partial U}$.

B.) If u_1, u_2 are solutions, $w = u_1 - u_2$ has $\int_U |w|^2 dx = 0$ by A), so $w = 0$ and $u_1 = u_2$.

Problem 5

A.) $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = -w^2 u - c^2(-|k|^2 u) = 0$

$\Rightarrow w^2 = |k|^2 \cdot c^2$

B.) $\{x \in \mathbb{R}^n \mid u(t,x) = e^{i\theta}\} = \{x \in \mathbb{R}^n \mid e^{ik\cdot x} = e^{i(wt+\theta)}\}$

$= \{x \in \mathbb{R}^n \mid k \cdot x = wt + \theta + 2\pi k \text{ for } k \in \mathbb{Z}\}$

C.) As t increases, it moves in direction k at speed c.

direction $\pm k \rightsquigarrow$ b/c $k \cdot x$

Speed $c \rightsquigarrow$ $|w| = |k| \cdot c \Rightarrow \frac{k}{|k|} \cdot x = \pm \underbrace{ct}_{\text{Speed } c} + \text{constant}$